Approximate solutions of the Schrödinger equation with energy-dependent screened Coulomb potential in D – dimensions

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ABSTRACT: Within the framework of the conventional Nikiforov-Uvarov method and a new form of Greene-Aldrich approximation scheme, we solved the Schrödinger equation with the energy-dependent screened Coulomb potential. Energy eigenvalues and energy eigenfunctions were obtained both approximately and numerically at different dimensions. The energy variations with different potential parameters, quantum numbers and energy slope parameter, respectively were also discussed graphically. The major finding of this research is the effect of the energy slope parameter on the energy spectra, which is seen in the existence of two simultaneous energy values for a particular quantum state. Our special cases also agree with the results obtained from literature, when the energy slope parameter is zero.
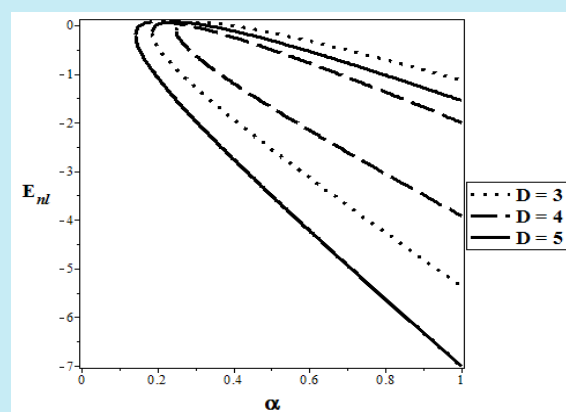
1. Introduction

Quantum mechanics came into existence many decades ago to salvage the failure of classical mechanics, not being able to explain some of the physical phenomena such as Compton effects, specific heat capacity, blackbody radiation. From this point of view, many theoretical physicists have been investigating the exact and approximate solutions of the Schrödinger equation for some potentials of physical interests[1,2]. The solutions of the Schrödinger equation play a vital role in many branches of modern physics and chemistry[3]. This is because it contains all the necessary information needed for the full description of a quantum state such as the probability density and entropy of the system[4].

The Schrödinger equation with many physical potentials model have been investigated in recent times with different analytical methods such as Nikiforov-Uvarov (NU) method[5-8], asymptotic iteration method (AIM)[9-14], supersymmetric quantum mechanics(SUSYQM)[15-18] among others[19-22]. One of such potential models is the screened Coulomb potential, which is given by Eq. 1[23].

$$V(r) = -\frac{Ae^{-\alpha r}}{r} \qquad (1)$$

The screened Coulomb potential, also known as the Yukawa potential is greatly important, with applications cutting across nuclear Physics and Condensed-matter Physics[23]. Here, its usage is involved in short-ranged interactions[24-26]. The screened-Coulomb potential is known to be the

potential of a charged particle in a weakly non-ideal plasma. It also describes the charged particle effects in a sea of conduction electrons in solid-state physics[27].

An approximate solution of the Schrödinger equation interacting with an inversely quadratic Yukawa potential has been obtained using SUSYQM[15], where the screened Coulomb potential was obtained as a special case by varying the potential strength. Also, an approximate analytical solution of the radial Schrödinger equation for the screened Coulomb potential has been obtained, with energy eigenvalues and its corresponding eigenfunctions computed in closed forms[28].

Several researchers have also devoted great attention to investigate the quantum systems of the energy dependence of different potentials[29-31]. Hassanabadi et al.[32,33] studied the exact solutions of D-dimensional Schrödinger and Klein-Gordon equations using the Nikiforov-Uvarov method. Also, Lombard et al.[34] investigated the wave equation energy-dependent potential for confined systems. Numerous applications of the energy-dependent potential of wave equations have been seen in the spectrum of confined systems and heavy quark confinement in nuclear and molecular physics[35,36]. Recently, Budaca[37] studied an energy-dependent Coulomb-like potential within the framework of Bohr Hamiltonian. The author further reported that the energy dependence on the coupling constant of the potential drastically changes the analytical properties of wave function and the corresponding eigenvalues of the system. Also, Boumali and Labidi[38] investigated the Shannon and Fisher information in the Klein-Gordon equation with energy-dependent potential.

In this research, we seek to investigate the influence of the energy-dependent screened Coulomb potential defined as in Eq. 2,

$$V(r, E_{n\ell}) = -\frac{A(1+gE_{n\ell})e^{-\alpha r}}{r} \tag{2}$$

where g is the energy slope parameter, A is the depth of the potential, and α is the range of the potential. The effects of the energy dependence on the screened Coulomb potential have not been considered before in any literature, to the best of our knowledge. It can be deduced that when $g = 0$, the potential of Eq. 2 reduces to the Screened Coulomb potential. When $g = 0$ as $\alpha \to 0$, the potential of Eq. 2 reduces to the Coulomb potential. Using the conventional NU method, we will derive the ℓ−wave bound state solutions and their eigenfunctions of the Schrödinger equation for the energy-dependent screened Coulomb potential, both analytically and numerically. Special cases are also considered and our results are compared with existing literature for confirmation sake.

The organization of this work is as follows: In section 2, we determine the eigensolutions of the energy-dependent screened Coulomb potential by employing a new form of Greene-Aldrich approximation scheme and Nikiforov-Uvarov method. Section 3 is devoted to discuss the results obtained and compare to results in relevant literature. The conclusion of the work is presented in section 4.

2. Bound state solution of the energy-dependent screened Coulomb potential

The radial part of the Schrödinger equation in D-dimension[39] is given by Eq. 3.

$$\psi''(r) + \frac{2\mu}{\hbar^2}(E_{n\ell} - V(r))\psi(r) + \frac{1}{r^2}\left[\frac{(D-1)(D-3)}{4} + \ell(\ell + D - 2)\right]\psi(r) = 0 \tag{3}$$

where μ is the reduced mass, $E_{n\ell}$ is the non-relativistic energy eigenvalues to be determined. Substituting Eq. 2 into Eq. 3 gives Eq. 4.

$$\psi''(r) + \frac{2\mu}{\hbar^2}\left(E_{n\ell} + \frac{A(1+gE_{n\ell})e^{-\alpha r}}{r}\right)\psi(r) + \frac{1}{r^2}\left[\frac{(D-1)(D-3)}{4} + \ell(\ell + D - 2)\right]\psi(r) = 0 \tag{4}$$

In solving Eq. 4, we invoke a new form of Greene-Aldrich approximation scheme[23] to deal with the centrifugal term since $\ell \neq 0$. The approximations schemes are given by Eq. 5 and 6.

$$f_1(r) = \frac{1}{r^2} \approx \frac{4\alpha^2 e^{-2\alpha r}}{(1-e^{-2\alpha r})^2} = f_2(r) \tag{5}$$

$$\frac{1}{r^2} \approx \frac{4\alpha^2 e^{-2\alpha r}}{(1-e^{-2\alpha r})^2} + \frac{\alpha^2}{3} = f_3(r) \tag{6}$$

The plots explaining the rationality and validity of the above Eq. 5 and 6 and their expansions are given in Fig. 1(a–d). Figure 1a shows the plots of $f_1(r), f_2(r)$ and $f_3(r)$ as they vary with r, when the screening parameter α is taken to be 0.5. In addition, we employ $\frac{1}{r}$ from $f_1(r)$ as given in Eq. 7.

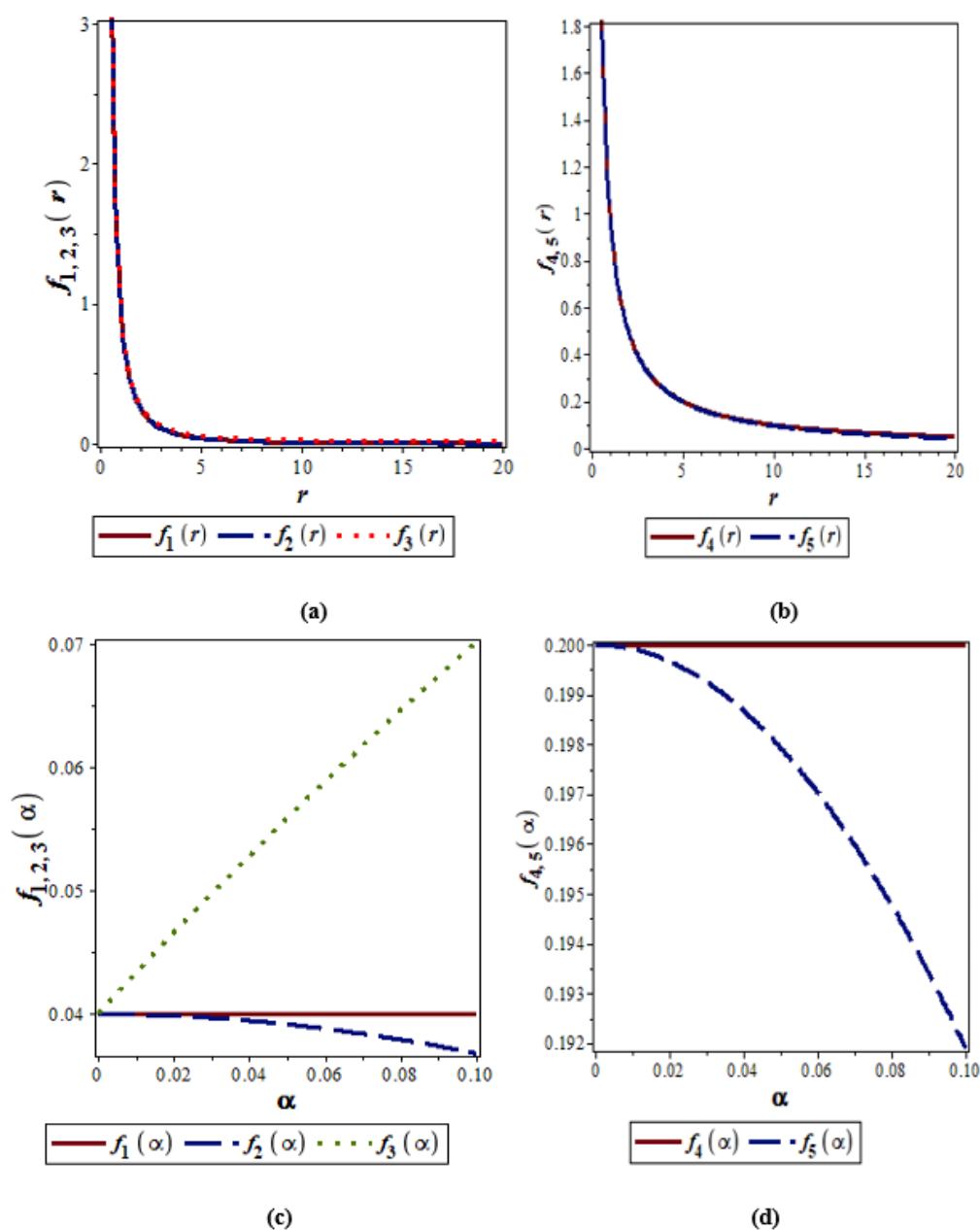$$f_4(r) = \frac{1}{r} \approx \frac{2\alpha e^{-\alpha r}}{1-e^{-2\alpha r}} \equiv f_5(r) \tag{7}$$

Figure 1. The plots of expressions (a) $f_1(r), f_2(r), f_3(r)$ as functions of r with $\alpha = 0.5$; (b) $f_4(r), f_5(r)$ as functions of r with $\alpha = 0.5$; (c) $f_1(\alpha), f_2(\alpha), f_3(\alpha)$ as functions of α with $r = 5$; (d) $f_4(\alpha), f_5(\alpha)$ as functions of α with $r = 5nm$.

Here, we have ignored the second expansion term $\frac{\alpha(1-e^{-2\alpha r})}{(12e^{-\alpha r})}$ when expanding Eq. 6. Furthermore, the rationality and validity of this expansion is given in Fig. 1b, with $\alpha = 0.5$. Considering the plots of the above expressions as function ns of the screening parameter α as shown in Figs. 1c and 1d, we observe that the approximation $f_3(\alpha)$ corresponds to that of $f_1(\alpha)$ when $\alpha \leq 0.05$. As such, we see that the approximation $f_3(\alpha)$ is better than that of $f_2(\alpha)$.

With the above approximation schemes, Eq. 4 becomes Eq. 8.

$$\psi''(r) + \left[\frac{2\mu E_{n\ell}}{\hbar^2} + \frac{4\mu A(1+gE_{n\ell})e^{-2\alpha r}}{\hbar^2(1-e^{-2\alpha r})} - \frac{4\alpha^2\gamma e^{-2\alpha r}}{\hbar^2(1-e^{-2\alpha r})^2} - \frac{\gamma\alpha^2}{3}\right]\psi(r) = 0 \tag{8}$$

where γ is given by Eq. 9.

$$\gamma = \left[\frac{(D-1)(D-3)}{4} + \ell(\ell + D - 2)\right] \tag{9}$$

By using the coordinate transformation of Eq. 10.

$$z = e^{-2\alpha r} \tag{10}$$

Eq. 8 becomes the differential equation of the form given in Eq. 11.

$$\psi''(z) + \frac{(1-z)}{z(1-z)}\psi'(z) + \frac{1}{z^2(1-z)^2}[-(\varepsilon^2 + \beta^2)z^2 + (2\varepsilon^2 + \beta^2 - \gamma)z - \varepsilon^2]\psi(z) = 0 \tag{11}$$

where ε^2 is given by Eq. 12.

$$\varepsilon^2 = -\left(\frac{\mu E_{n\ell}}{2\hbar^2\alpha^2} - \frac{\gamma}{12}\right); \beta^2 = \frac{\mu A(1+gE_{n\ell})}{\hbar^2\alpha} \tag{12}$$

By comparing Eq. 11 and Eq. A1 (see Appendix), we have the following parameters, Eq. 13:

$$\tilde{\tau}(z) = 1 - z$$
$$\sigma(z) = z(1 - z)$$
$$\tilde{\sigma}(z) = -(\varepsilon^2 + \beta^2)z^2 + (2\varepsilon^2 + \beta^2 - \gamma)z - \varepsilon^2 \tag{13}$$

Substituting Eq. 13 into Eq. A8 (see Appendix), we get $\pi(z)$, Eq. 14:

$$\pi(z) = -\frac{z}{2} \pm \sqrt{(a - k)z^2 + (k + b)z + c} \tag{14}$$

where a, b and c are given by Eq. 15.

$$a = \frac{1}{4} + \varepsilon^2 + \beta^2$$

$$b = \gamma - 2\varepsilon^2 - \beta^2 \tag{15}$$

$$c = \varepsilon^2$$

We can obtain the constant k, by expressing the discriminant under the square root of Eq. 14 to be equal to zero. As such, we have Eq. 16

$$k_\pm = -(\gamma - \beta^2) \pm 2\sqrt{\varepsilon^2}\sqrt{\left(\frac{1}{4} + \gamma\right)} \tag{16}$$

Substituting Eq. 16 into Eq. 14 yields Eq. 17:

$$\pi(z) = -\frac{z}{2} \pm \begin{cases} \left(\left(\sqrt{\varepsilon^2} - \sqrt{\frac{1}{4} + \gamma}\right)z - \sqrt{\varepsilon^2}; for\, k_+ = -(\gamma - \beta^2) + 2\sqrt{\varepsilon^2}\sqrt{\left(\frac{1}{4} + \gamma\right)} \right. \\ \left. \left(\sqrt{\varepsilon^2} - \sqrt{\frac{1}{4} + \gamma}\right)z + \sqrt{\varepsilon^2}; for\, k_- = -(\gamma - \beta^2) - 2\sqrt{\varepsilon^2}\sqrt{\left(\frac{1}{4} + \gamma\right)} \right. \end{cases} \tag{17}$$

According to NU method[40], we choose the expression $\pi(z)_-$ which the function $\tau(z)$ has a negative derivative. This is given by Eq. 18

$$\pi(z)_- = -\left(\frac{1}{2} + \sqrt{\varepsilon^2} + \sqrt{\frac{1}{4} + \gamma}\right)z - \sqrt{\varepsilon^2} \tag{18}$$

with $\tau(z)$ being obtained with Eq. 19.

$$\tau(z) = 1 - 2\sqrt{\varepsilon^2} - 2\left(1 + 2\sqrt{\varepsilon^2} + \sqrt{\frac{1}{4} + \gamma}\right)z \tag{19}$$

By recalling Eq. A9 (see Appendix), we define the constant λ (Eq. 20) as

$$\lambda = -(\gamma - \beta^2) - 2\sqrt{\varepsilon^2}\sqrt{\frac{1}{4} + \gamma} - \left(\frac{1}{2} + \sqrt{\varepsilon^2} + \sqrt{\frac{1}{4} + \gamma}\right) \tag{20}$$

Substituting Eq. 20 into Eq. A10 (see Appendix) and carrying out algebraic simplifications, where $\tau^{'}(z)$ is given by Eq. 21

$$\tau^{'}(z) = -2\left(1 + \sqrt{\varepsilon^2} + \sqrt{\frac{1}{4} + \gamma}\right) \tag{21}$$

and $\sigma^{''}(z)$, Eq. 22

$$\sigma''(z) = -2 \tag{22}$$

we obtain Eq. 23

$$\varepsilon^2 = \frac{1}{4}\left[\frac{(n+\chi)^2 - \beta^2}{(n+\chi)}\right]^2 \tag{23}$$

where χ is given by Eq. 24

$$\chi = \frac{1}{2}\left(1 + \sqrt{1 + 4\gamma}\right) \tag{24}$$

Substituting Eq. 12 into Eq. 23 yields a complicated transcendental energy eigenvalue equation of the energy dependent screened Coulomb potential in D-dimensions as (Eq. 25):

$$\left[(n+\chi) - \frac{\mu A(1 + gE_{n\ell})}{\hbar^2\alpha(n+\chi)}\right]^2 + \frac{2\mu E_{n\ell}}{\hbar^2\alpha^2} - \frac{\gamma}{3} = 0 \tag{25}$$

Eq. 25 can also be expressed by Eq. 26.

$$E_{n\ell} = \frac{-\hbar^2 \alpha^2}{2\mu} \left[\left((n+\chi) - \frac{\mu A(1+gE_{n\ell})}{\hbar^2 \alpha(n+\chi)} \right)^2 - \frac{\gamma}{3} \right] \tag{26}$$

where γ and χ are given in Eqs. 9 and 24, respectively.

To obtain the special case, we first rewrite Eq. 26 to the form of Eq. 27:

$$E_{n\ell} = \frac{-\hbar^2}{2\mu} \left[\left(\alpha(n+\chi) - \frac{\mu A(1+gE_{n\ell})}{\hbar^2(n+\chi)} \right)^2 - \frac{\alpha^2 \gamma}{3} \right] \tag{27}$$

As $\alpha \to 0\, and\, g \to 0$, Eq. 2 reduces to the standard Coulomb potential of the form of Eq. 28.

$$V(r) = -\frac{A}{r} \tag{28}$$

Setting the parameters $D = 3$ as $\alpha \to 0\, and\, g \to 0$, we obtain the energy eigenvalue equation (Eq. 29):

$$E_{n\ell} = -\frac{\mu A^2}{2\hbar^2(n+\ell+1)^2} \tag{29}$$

This result is very consistent with the result obtained in Eq. 101 of Birkdemir *et al.*[41]. Also, taking the natural units ($\hbar^2 = \mu = 1$) and setting $D = 3\, and\, g = 0$, the energy eigenvalue expression of Eq. 26 can be reduced to Eq. 30.

$$E_{n\ell} = \frac{-\alpha^2}{2} \left[\left((n+\ell+1) - \frac{A}{\alpha(n+\ell+1)} \right)^2 - \frac{\ell(\ell+1)}{3} \right] \tag{30}$$

The result of Eq. 30 is consistent with the result obtained in Eq. 18 of Dong *et al.*[23].

To obtain the corresponding wave functions, we substitute $\pi(z)\_ and\, \sigma(z)$ from Eqs. 18 and 13, respectively into Eq. A4 (see Appendix) and solve the first-order differential equation. This gives Eq. 31.

$$\Phi(z) = z^{\sqrt{\varepsilon^2}}(1-z)^{\frac{1}{2}+\sqrt{\frac{1}{4}+\gamma}} \tag{31}$$

The weight function $\rho(z)$ from Eq. A6 (see Appendix) can be obtained Eq. 32

$$\rho(z) = z^{2\sqrt{\varepsilon^2}}(1-z)^{2\sqrt{\frac{1}{4}+\gamma}} \tag{32}$$

From the Rodrigues relation of Eq. A5 (see Appendix), we obtain Eq. 33 and Eq. 34.

$$y_n(z) = B_n z^{-2\sqrt{\varepsilon^2}}(1-z)^{-2\sqrt{\frac{1}{4}+\gamma}} \frac{d^n}{dz^n} \left[z^{n+2\sqrt{\varepsilon^2}}(1-z)^{n+2\sqrt{\frac{1}{4}+\gamma}} \right] \tag{33}$$

$$y_n(z) \equiv B_n P_n^{\left(2\sqrt{\varepsilon^2}, 2\sqrt{\frac{1}{4}+\gamma}\right)}(1-2z) \tag{34}$$

where $P_n^{(\theta,\vartheta)}$ is the Jacobi Polynomial.

Substituting $\Phi(z)\, and\, y_n(z)$ from Eqs. 31 and 34, respectively into Eq. A2 (see Appendix), we obtain Eq. 35:

$$\psi(z) = B_n z^{\sqrt{\varepsilon^2}} (1-z)^G P_n^{(2\sqrt{\varepsilon^2}, 2G-1)}(1-2z) \tag{35}$$

where G is given by Eq. 36.

$$G = \frac{1}{2} + \sqrt{\frac{1}{4} + \gamma} \tag{36}$$

From the definition of the Jacobi Polynomials[42] results Eq. 37.

$$P_n^{(\theta,\vartheta)}(\omega) = \frac{\Gamma(n+\theta+1)}{n!\,\Gamma(\theta+1)} \, {}_2F_1\left(-n, \theta+\vartheta+n+1, \theta+1; \frac{1-\omega}{2}\right) \tag{37}$$

In terms of hypergeometric Polynomials, Eq. 35 can be written as Eq. 38.

$$\psi(z) = B_n z^{\sqrt{\varepsilon^2}} (1-z)^G \frac{\Gamma\left(n+2\sqrt{\varepsilon^2}+1\right)}{n!\,\Gamma\left(2\sqrt{\varepsilon^2}+1\right)} \, {}_2F_1\left(-n, \, 2\sqrt{\varepsilon^2}+2G+n, \, 2\sqrt{\varepsilon^2}+1; \, z\right). \tag{38}$$

3. Results and Discussion

In this study, the energy eigenvalues for the energy-dependent screened Coulomb potential as they vary with the screening parameter were computed for different quantum states as shown in Tab. 1. The existence of the energy slope parameter in Eq. 26 results in two different energy spectra for a particular quantum state and a specific screening parameter. The screening parameter varies in an inverse version with the duo energy spectra. The energy eigenvalues for the energy-dependent screened Coulomb potential as a function of screening parameters in higher dimensions were also computed, as shown in Tabs. 2 and 3, respectively. By employing Eq. 30, we have also computed the energy eigenvalues in three dimensions and in the absence of the energy slope parameter, as shown in Tab. 4. For the two different potential depths considered, we obtained the bound state energy eigenvalues for different quantum states of **2p-4f**, as they vary with different screening parameters. It can be seen that the energy eigenvalues for the different potential depths decrease as the screening parameter increases, at each quantum state. Our analytical result of Eq. 30 and its corresponding numerical results of Tab. 4 are very consistent with the results obtained by Dong et al.[23]. We have also computed the energy eigenvalues of Eq. 26 for higher dimensions, as shown in Tabs. 5 and 6, respectively. The level of effects imposed by the potential parameters of Eq. 2 on the energy eigenvalues of Eq. 26 are shown in Figs. 2-7. For different dimensions, the energy slope parameter causes an interwoven interaction of the curves, as compared to the situation when the energy slope parameter is zero. Figs. 8 and 9 show the variation of energy eigenvalues for the energy-dependent screened Coulomb potential with the different quantum numbers, for different values energy slope parameter considered. The trend of the relationship shows that the energy eigenvalues decrease as the quantum numbers increase. In Fig. 10, we also plotted the variation of the energy eigenvalues with the energy slope parameter for different dimensions.

Table 1. Eigenvalues $(E_{n\ell})$ for the energy-dependent screened Coulomb potential as a function of the parameter α for **2p-4f** states, with $D = 3$.

States	α	$E_{n\ell}(g = -1, A = 2)$	$E_{n\ell}(g = 1, A = 2)$
	0.025	$-0.05000000000, -0.9484636700$	$-0.2469390303, -3.653060970$
$2p$	0.050	$-0.1000000000, -0.8934950102$	$-0.2261820091, -3.573817991$
	0.075	$-0.1499999989, -0.8344159637$	$-0.2056916349, -3.494308363$

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	0.025	−0.3282430205, − 2.396756978	−0.1279838429, − 6.147016156
3p	0.050	−0.2185168124, − 2.731483190	−0.1003205303, − 5.949679472
	0.075	−0.1419248550, − 3.033075144	−0.07486233817, − 5.750137661
	0.025	−0.3273369696, − 2.397663029	−0.1276723472, − 6.147327652
3d	0.050	−0.2155358280, − 2.734464174	−0.09903861948, − 5.950961383
	0.075	−0.1360998141, − 3.038900185	−0.07189047036, − 5.753109529
	0.025	−0.1013443775, − 6.298655622	−0.06696010299, − 9.533039897
4p	0.050	−0.05236401820, − 6.747635982	−0.03856747648, − 9.161432524
	0.075	−0.02019553609, − 7.179804460	−0.01650824116, − 8.783491756
	0.025	−0.1008065565, − 6.299193443	−0.06660798158, − 9.533392018
4d	0.050	−0.05037315511, − 6.749626845	−0.03710618138, − 9.162893819
	0.075	−0.01600781253, − 7.183992184	−0.01308764621, − 8.786912351
	0.025	−0.1000000000, − 6.300000000	−0.06607984858, − 9.533920151
4f	0.050	−0.04738907715, − 6.752610923	−0.03491511579, − 9.165084884
	0.075	−0.009735385820, − 7.190264611	−0.007961748818, − 8.792038248

Table 2. Eigenvalues $(E_{n\ell})$ for the energy-dependent screened Coulomb potential as a function of the parameter α for **2p-4f** states, with $D = 4$.

States	α	$E_{n\ell}(g = -1, A = 2)$	$E_{n\ell}(g = 1, A = 2)$
	0.025	−0.6406250000, − 0.6619912551	−0.1771068474, − 4.791643153
2p	0.050	−0.7187500000, − 0.4363825014	−0.1516973143, − 4.660802686
	0.075	−0.5523050676, − 1.041444930	−0.1269955847, − 4.529254413
	0.025	−0.1676529658, − 4.263597033	−0.09251657939, − 7.726233419
3p	0.050	−0.1017568786, − 4.635743120	−0.06332496524, − 7.449175033
	0.075	−0.05426252476, − 4.989487474	−0.03776841503, − 7.168481584
	0.025	−0.1668742692, − 4.264375729	−0.09209870569, − 7.726651293
3d	0.050	−0.09894423101, − 4.638555767	−0.06159768496, − 7.450902314
	0.075	−0.04845181230, − 4.995298186	−0.03374430925, − 7.172505689
	0.025	−0.06465429287, − 8.566595705	−0.04786732380, − 11.57088267
4p	0.050	−0.02501719398, − 9.112482806	−0.02055263540, − 11.09194736
	0.075	−0.002313419520, − 9.641436577	−0.002103392985, − 10.60414660
	0.025	−0.06403407537, − 8.567215922	−0.04740969809, − 11.57134030
4d	0.050	−0.02269659485, − 9.114803405	−0.01864771539, − 11.09385228
	0.075	0.002607849543, − 9.646357846	0.002371302471, − 10.60862130
	0.025	−0.06316592284, − 8.568084075	−0.04676908315, − 11.57198091
4f	0.050	−0.01944974474, − 9.118050255	−0.01598192774, − 11.09651807
	0.075	−0.009489204657, − 9.653239201	−0.008629543278, − 10.61487954

Table 3. Eigenvalues $(E_{n\ell})$ for the energy-dependent screened Coulomb potential as a function of the parameter α for **2p-4f** states, with $D = 5$.

States	α	$E_{n\ell}(g = -1, A = 2)$	$E_{n\ell}(g = 1, A = 2)$
	0.025	−0.3273369696, − 2.397663029	−0.1276723472, − 6.147327652
2p	0.050	−0.2155358280, − 2.734464174	−0.09903861948, − 5.950961383
	0.075	−0.1360998141, − 3.038900185	−0.07189047036, − 5.753109529
	0.025	−0.1008065565, − 6.299193443	−0.06660798158, − 9.533392018
3p	0.050	−0.05037315511, − 6.749626845	−0.03710618138, − 9.162893819
	0.075	−0.01600781253, − 7.183992184	−0.01308764621, − 8.786912351
	0.025	−0.1000000000, − 6.300000000	−0.06607984858, − 9.533920151
3d	0.050	−0.04738907715, − 6.752610923	−0.03491511579, − 9.165084884
	0.075	−0.009735385820, − 7.190264611	−0.007961748818, − 8.792038248
	0.025	−0.4194183076, − 11.08305817	−0.03358353903, − 13.84141646
4p	0.050	−0.009315896675, − 11.74068410	−0.008259866067, − 13.24174013
	0.075	0.005363836706, − 12.38036383	0.005257711402, − 12.63025771
	0.025	−0.04123429366, − 11.08376571	−0.03301776018, − 13.84198224
4d	0.050	−0.006652702848, − 11.74334730	−0.005898852562, − 13.24410115
	0.075	0.01103813411, − 12.38603813	0.01081993497, − 12.63581993
	0.025	−0.04029105188, − 11.08470895	−0.03226346048, − 13.84273654
4f	0.050	−0.003103656681, − 11.74689634	−0.002752143970, − 13.24724786
	0.075	0.01859579365, − 12.39359579	0.01822863105, − 12.64322863

Table 4. Eigenvalues $(E_{n\ell})$ of Eq. 29 as a function of the parameter α for **2p-4f** states, with $D = 3$ and $g = 0$.

States	α	$E_{n\ell}(A = 2.0)$	$E_{n\ell}(A = 5.0)$
2p	0.025	−0.4510416666	−3.001041666
	0.050	−0.4041666666	−2.879166666
	0.075	−0.3593750004	−2.759375001
3p	0.025	−0.1748263890	−1.266493056
	0.050	−0.1326388888	−1.149305555
	0.075	−0.09565972225	−1.037326388
3d	0.025	−0.1744097223	−1.266076389
	0.050	−0.1309722221	−1.147638888
	0.075	−0.09190972225	−1.033576388
4p	0.025	−0.07979166665	−0.6610416665
	0.050	−0.04416666666	−0.5504166665
	0.075	−0.01812500002	−0.4493750000
4d	0.025	−0.07937500000	−0.6606250000
	0.050	−0.04250000000	−0.5487500000
	0.075	−0.01437500002	−0.4456250002
4f	0.025	−0.07875000000	−0.6600000000
	0.050	−0.04000000000	−0.5462500000
	0.075	−0.008750000020	−0.4400000002

Table 5. Eigenvalues $(E_{n\ell})$ of Eq. 25 as a function of the parameter α for **2p-4f** states, with $D = 4$ and $g = 0$.

States	α	$E_{n\ell}(A = 2.0)$	$E_{n\ell}(A = 5.0)$
2p	0.025	−0.2715625000	−1.876562500
	0.050	−0.2262500000	−1.756250000
	0.075	−0.1840625002	−1.639062500
3p	0.025	−0.1167028062	−0.8988456630
	0.050	−0.07701530615	−0.7841581630
	0.075	−0.04420280616	−0.6763456635
3d	0.025	−0.1161819728	−0.8983248295
	0.050	−0.07493197280	−0.7820748300
	0.075	−0.03951530614	−0.6716581635
4p	0.025	−0.05470293210	−0.4982214506
	0.050	−0.02251543210	−0.3910339505
	0.075	−0.002202932106	−0.2957214510
4d	0.025	−0.05418209880	−0.4977006172
	0.050	−0.02043209876	−0.3889506171
	0.075	0.002484567894	−0.2910339509
4f	0.025	−0.05345293210	−0.4969714506
	0.050	−0.01751543210	−0.3860339505
	0.075	0.009047067895	−0.2844714510

Table 6. Eigenvalues $(E_{n\ell})$ of Eq. 26 as a function of the parameter α for **2p-4f** states, with $D = 5$ and $g = 0$.

States	α	$E_{n\ell}(A = 2.0)$	$E_{n\ell}(A = 5.0)$
2p	0.025	−0.1744097223	−1.266076389
	0.050	−0.1309722221	−1.147638888
	0.075	−0.09190972225	−1.033576388
3p	0.025	−0.07937500000	−0.6606250000
	0.050	−0.04250000000	−0.5487500000
	0.075	−0.01437500002	−0.4456250002
3d	0.025	−0.07875000000	−0.6600000000
	0.050	−0.04000000000	−0.5462500000
	0.075	−0.008750000020	−0.4400000002
4p	0.025	−0.03718750000	−0.3821875000
	0.050	−0.008750000000	−0.2787500000
	0.075	0.005312500000	−0.1896874998
4d	0.025	−0.03656250000	−0.3815625000
	0.050	−0.006250000000	−0.2762500000
	0.075	0.01093750000	−0.1840624998

Continue...

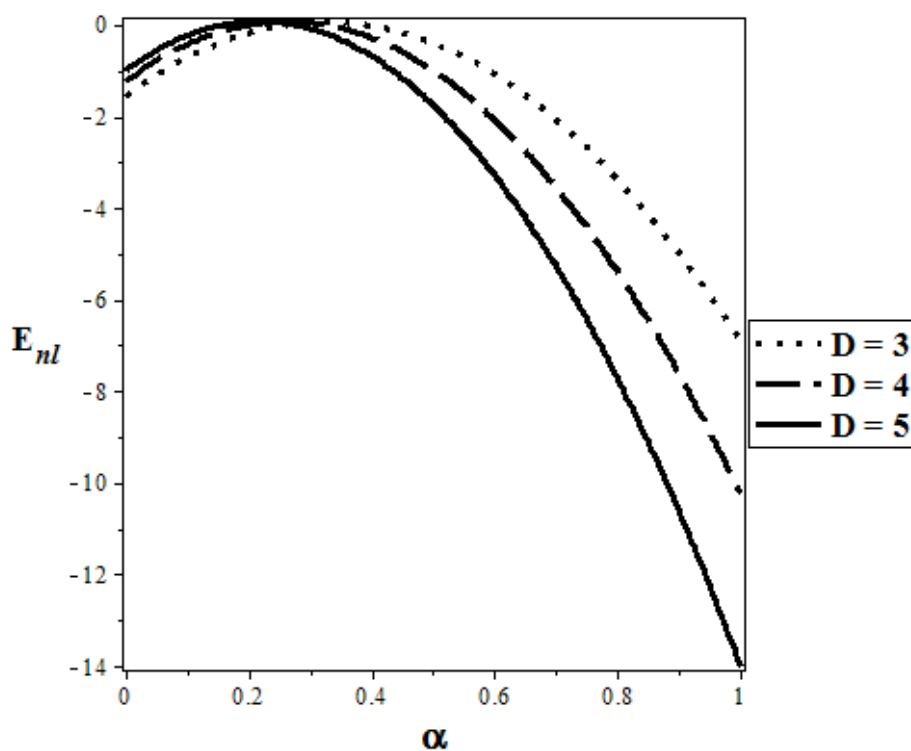|	0.025	−0.03572916666	−0.3807291666
$4f$	0.050	−0.002916666666	−0.2729166666
	0.075	0.01843750000	−0.1765624998

Figure 2. Energy Eigenvalue Variation with α for different dimensions with
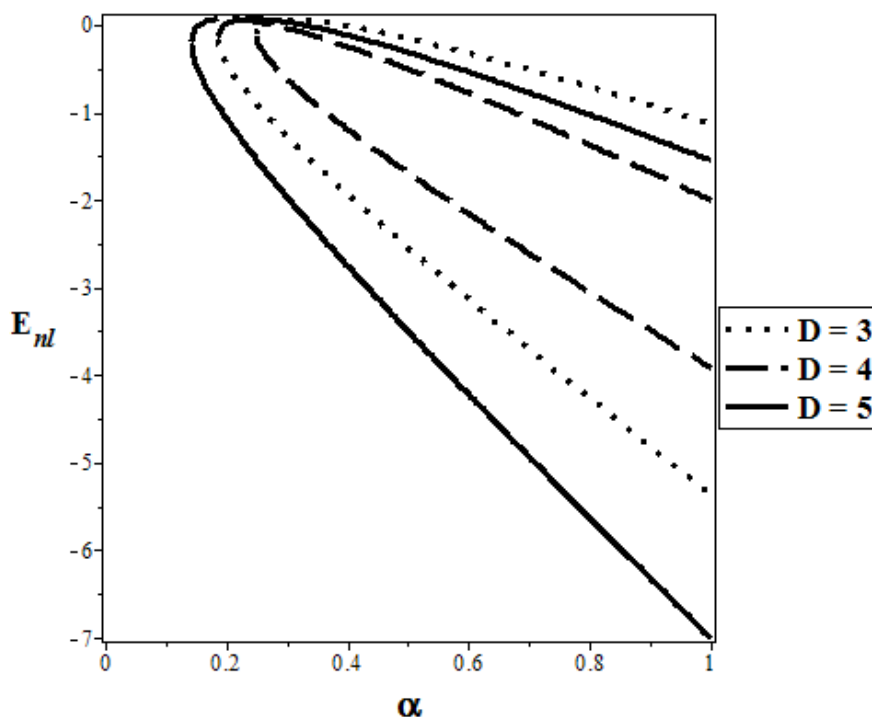$g = 0, A = 5, n = 2, \ell = 1$.

Figure 3. Energy Eigenvalue variation with α for different dimensions with
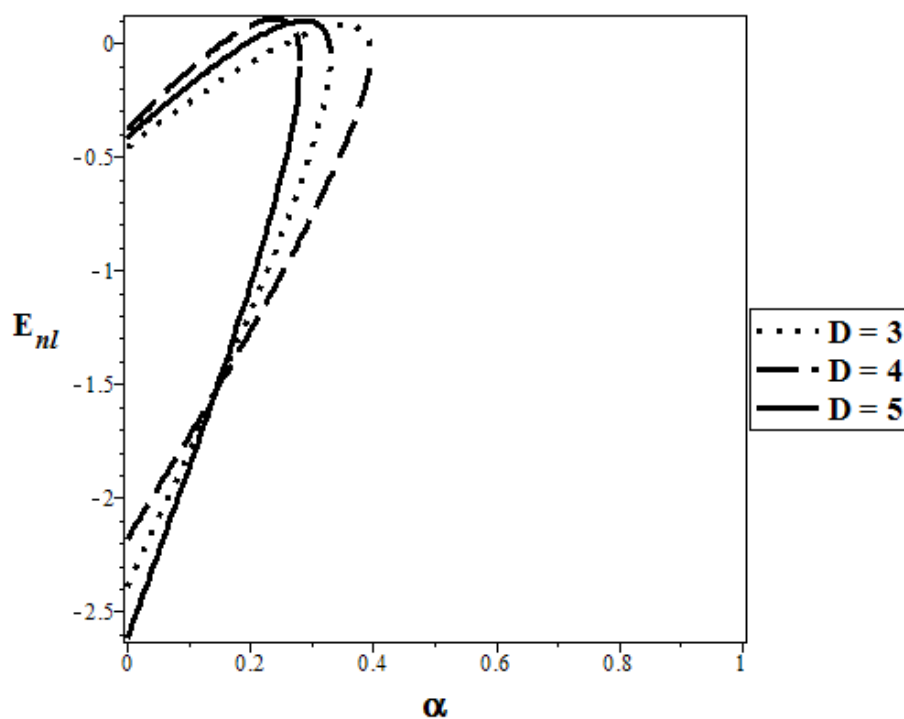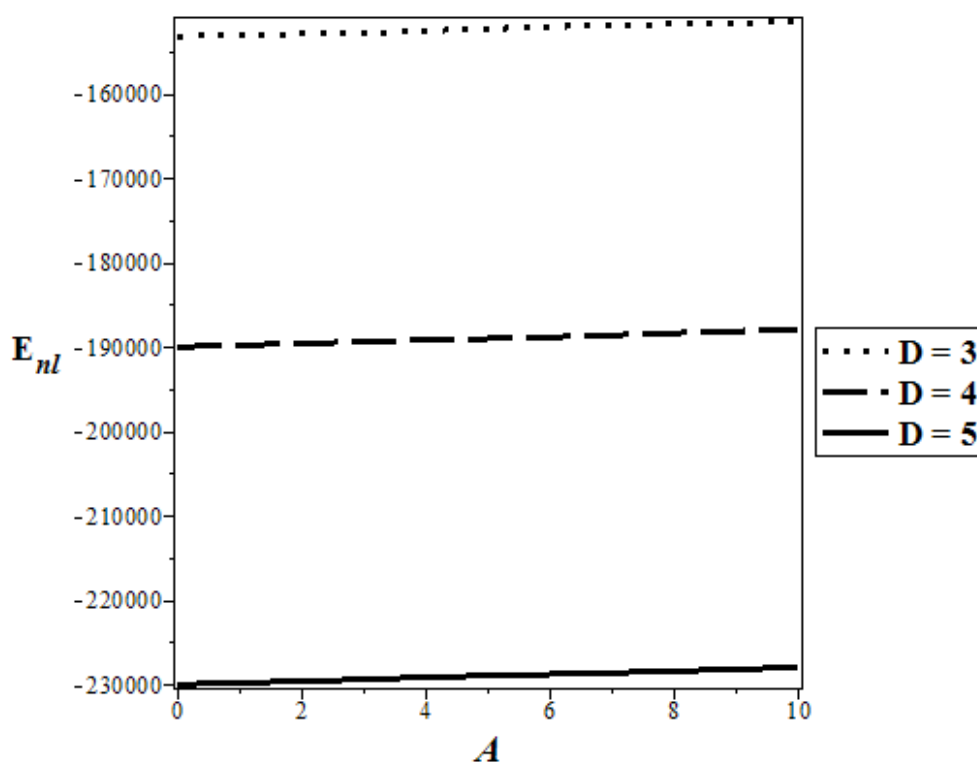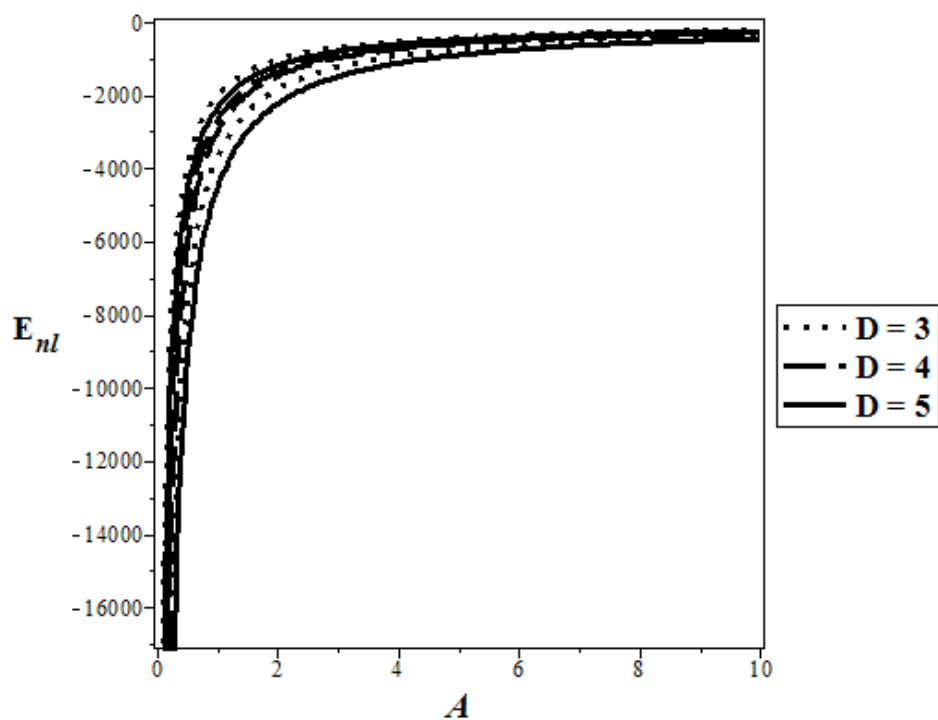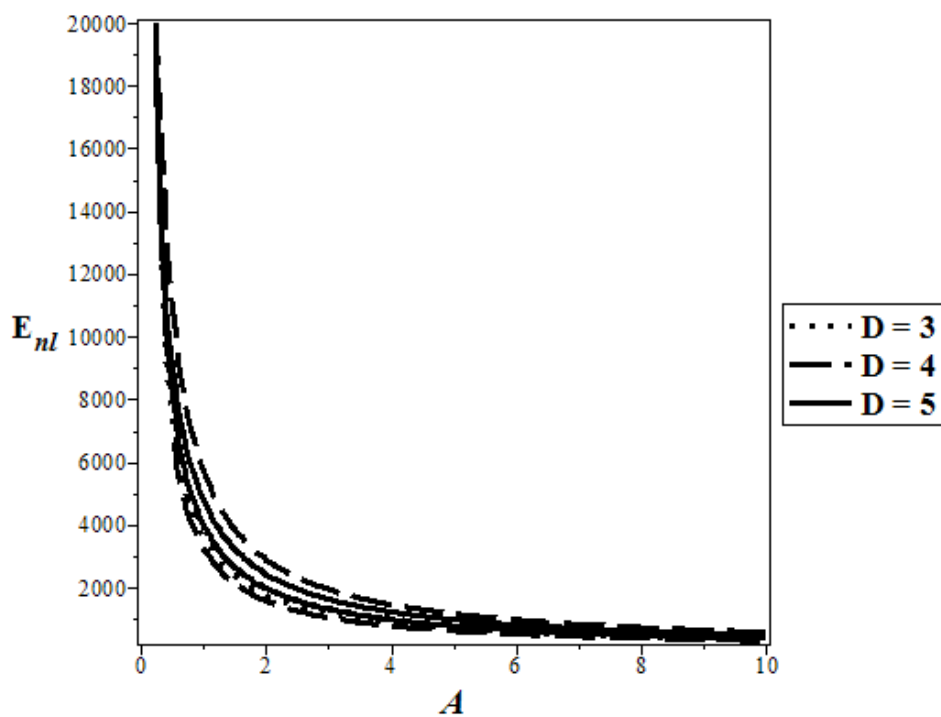$g = -1, A = 5, n = 2, \ell = 1$.

Figure 4. Energy Eigenvalue Variation with α for different dimensions with $g = 1, A = 5, n = 2, \ell = 1$.

Figure 5. Energy Eigenvalue Variation with A for different dimensions with $g = 0, \alpha = 100, n = 2, \ell = 1$.

Figure 6. Energy Eigenvalue Variation with *A* for different dimensions with
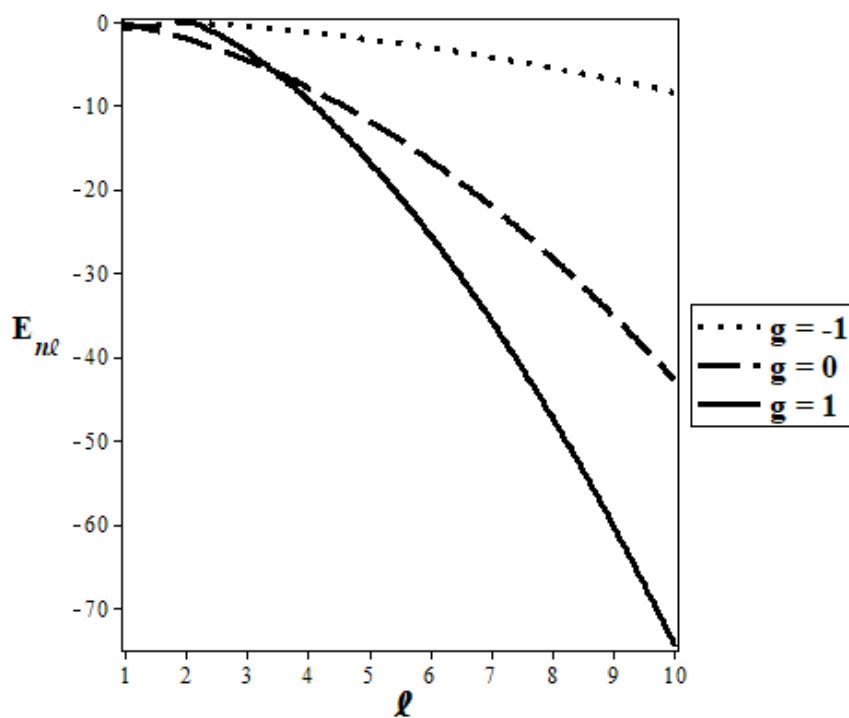$g = -1, \alpha = 100, n = 2, \ell = 1$.

Figure 7. Energy Eigenvalue Variation with *A* for different dimensions with
$g = 1, \alpha = 100, n = 2, \ell = 1$.

Figure 8. Energy Eigenvalue Variation with ℓ for different values of g with $D = 3, \alpha = 1, A = 5, n = 0$.
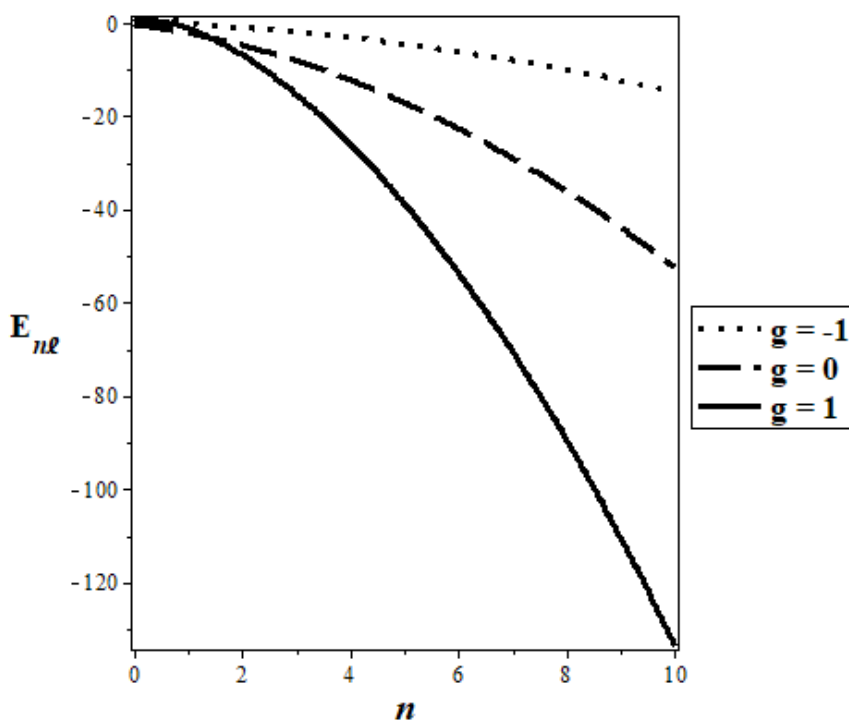

Figure 9. Energy Eigenvalue Variation with n for different values of g with $D = 3, \alpha = 1, A = 5, \ell = 1$.
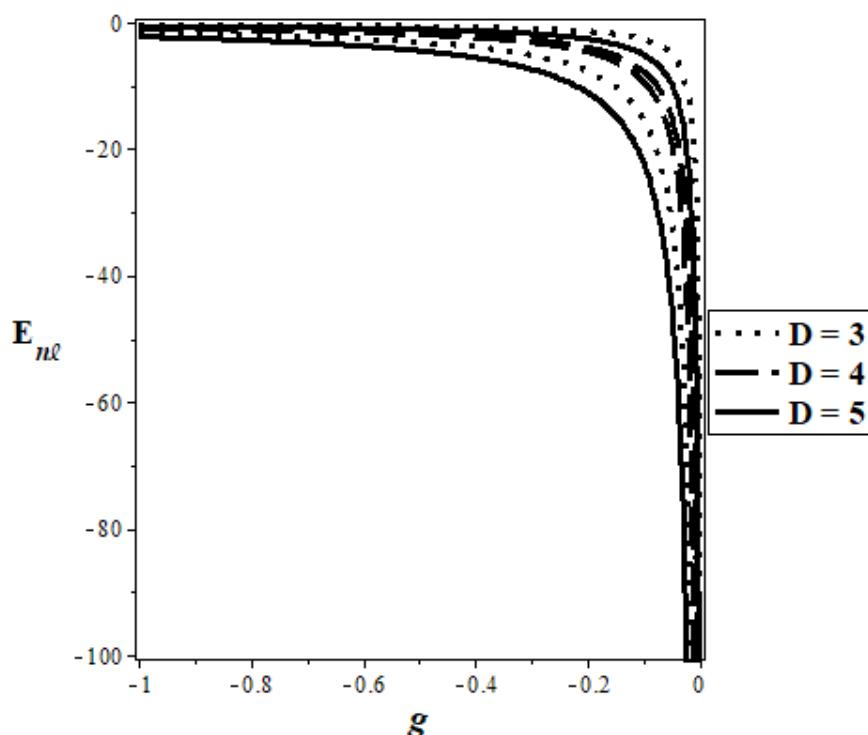
Figure 10. Energy Eigenvalue Variation with g for different dimensions, with
$\alpha = 10, A = 100, n = 2, \ell = 1$

4. Conclusions

We have solved the Schrödinger equation with the energy-dependent screened Coulomb potential in higher dimensions. With the use of the conventional Nikiforov-Uvarov method and a new form of Greene-Aldrich approximation, the energy eigenvalues and its corresponding eigenfunctions were obtained approximately. Numerical values of the energy eigenvalues were also obtained with natural units, in three dimensions. We have also elucidated the variation of these energy spectra with different potential parameters and the energy slope parameter. The effect of the energy slope parameter is clearly seen in the existence of duo energy spectra, as compared to when the energy slope parameter is diminished. Special cases have been deduced, and these results agree perfectly with available literature.

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Appendix: Review of Nikiforov-Uvarov (NU) method

According to Nikiforov and Uvarov[40], the NU method transforms Schrödinger-like equations into a second order differential equation using a coordinate transformation $z = z(r)$, which is given by Eq. A1.

$$\psi'' \frac{\tilde{\tau}(z)}{\sigma(z)}' \frac{\tilde{\sigma}(z)}{\sigma^2(z)} \tag{A1}$$

Here, $\tilde{\sigma}(z), \sigma(z)$ are polynomials of at most second degree, and $\tilde{\tau}(z)$ is a first-degree polynomial. By employing the transformation (Eq. A2),

$$\psi(z) = \Phi(z)y_n(z) \tag{A2}$$

we obtain the exact solution of Eq. A1 in a form of hypergeometric-type equation given by Eq. A3.

$$\sigma(z)y_n''(z) + \tau(z)y_n'(z) + \lambda y_n(z) = 0 \tag{A3}$$

Let us define the logarithm derivative function $\Phi(z)$ as[40] (Eq. A4).

$$\frac{\Phi'(z)}{\Phi(z)} = \frac{\pi(z)}{\sigma(z)} \tag{A4}$$

where $\pi(z)$ is at most a first-degree polynomial. The second part of $\psi(z)$ being $y_n(z)$ in Eq. A2, is the hypergeometric function with its polynomial solution given by Rodrigues relation (Eq. A5)

$$y_n(z) = \frac{B_n}{\rho(z)} \frac{d^n}{dz^n}[\sigma^n(z)\rho(z)] \tag{A5}$$

Let us mention here that B_n is the normalization constant and $\rho(z)$ is the weight function which must satisfy the condition expressed by Eq. A6.

$$\frac{d}{dz}[\sigma(z)\rho(z)] = \tau(z)\rho(z) \tag{A6}$$

with Eq. A7

$$\tau(z) = \tilde{\tau}(z) + 2\pi(z) \tag{A7}$$

The eigenfunctions and eigenvalues can be obtained using the definition of the following function $\pi(z)$ and parameter λ as shown (Eq. A8):

$$\pi(z) = \frac{\sigma'(z) - \tilde{\tau}(z)}{2} \pm \sqrt{\left(\frac{\sigma'(z) - \tilde{\tau}(z)}{2}\right)^2 - \tilde{\sigma}(z) + k\sigma(z)} \tag{A8}$$

and (Eq. A9)

$$\lambda = k + \pi'(z) \tag{A9}$$

To obtain the value of k, we set the discriminant of the square root in Eq. A8 equal to zero. As such, the new eigenvalue equation (Eq. A10) is obtained as

$$\lambda + n\tau'(z) + \frac{n(n-1)}{2}\sigma''(z) = 0, \ (n = 0,1,2,...) \tag{A10}$$